International Journal of Theoretical Physics, Vol. 45, No. 6, June 2006 (© 2006)
DOI: 10.1007/s10773-006-9115-2

Quantum Statistic Entropy of Three-Dimensional BTZ Black Hole

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Received September 26, 2005; accepted March 5, 2006
Published Online: June 7, 2006

Using the new equation of state density motivated by the generalized uncertainty relation in the quantum gravity, we investigate entropy of a black line on the background of the three-dimensional BTZ. In our calculation, we need not introduce cutoff and can remove the divergent term in the original brick-wall method via the new equation of state density. And it is obtained that the entropy of the black line is proportional to the area of the horizon (perimeter). Further it is shown the entropy of black line is the entropy of quantum state on the surface of horizon (perimeter). The black line entropy is the intrinsic property of the black hole. The entropy is a quantum effect. By using quantum statistical method, we directly obtain the partition function of Bose field and fermi field on the background of the black line. The difficulty to solve wave equation of various particles is avoided. We offer a new simple and direct way for calculating the entropy of various spacetime black holes (black plane, black line and black column).

KEY WORDS: entropy of BTZ black hole; quantum statistics; generalized uncertainty relation.

PACS: 04.20.Dw; 97.60.Lf.

1. INTRODUCTION

Entropy of the black hole is one of the important subjects in theoretical physics. Since entropy has statistical meaning, the understanding of entropy involves the sense of the microscopic essence of the black hole. Fully understanding of it needs a good quantum gravitation theory. However, at present the work of it is not satisfying. How to measure the microscopic state of black hole by entropy is not understood very well. The statistical origin of the black hole is not solved yet (Liberati, 1997). On the other hand, Since Bekenstein and Hawking derived

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that the entropy of the black entropy is proportional to the area of the horizon (Bekenstein, 1973; Hawking, 1975; Gibbons and Hawking, 1977), investigating the thermal property (Zhao *et al.*, 2001) is one important subject. Especially there have been a number of efforts in the past several years aimed at the entropy of black hole. Many methods of calculating entropy have emerged (Hochberg *et al.*, 1993; Padmanaban, 1989; Li and Zhao, 2000; Hooft, 1985; Cognola and Lecca, 1998; Cai *et al.*, 1998). One frequently used method is the brick-wall method advanced by G't Hooft (1985). This method is used to study the statistical properties of the free scalar field and fermi field in asymptotically flat spacetime under various spherical coordinates (Solodukhin, 1995; Zhao *et al.*, 2001; Carlip, 1995; Strominger, 1998; Jing and Yan, 2000) and it is found that the general expression of the black hole entropy consists of a term which is proportional to the area of its horizon and a divergent logarithmic term which is not proportional to the area of the horizon. However it is doubted that, (1) to obtain that entropy of the black hole is proportional to the area of the horizon we must introduce cutoff; (2) state density diverges near the horizon; (3) to obtain that the entropy of a black hole is proportional to the area of its horizon the logarithmic term is left out and L^3 (L is a distance between event horizon and a point outside horizon at infinite distance , that is $L >> r_+$) is considered as the contribution of distant vacuum surrounding the system; (4) it is complicated to derive the wave function of the scalar or Dirac field on the background of various black hole by WKB approximation; (5) how to compute entropies of high-dimensional spacetime, lower-spacetime and non-asymptotic spacetime. The above problems with the original brick-wall method are unnatural and insuperable.

In the other hand, there has been much interest recently in the region of lower-dimensional gravitation theory. Recently, the research given for two-dimensional black hole thermodynamics shows that entropy satisfies area relation and the second law of thermodynamics (Myers, 1994; Russo, 1995; Hayward, 1995; Gao and Shen, 2003). Banados, Teitelboim and Zanelli (BTZ) derived a black hole solution in the three-dimensional gravitation theory. This solution is depicted by mass and charge. The solution is asymptotically anti-de Sitter but not is asymptotically flat (Banados *et al.*, 1992). It is similar to the two-dimensional case. This solution does not contain the complicated dynamics freedom of four-dimensional Einstein gravitation theory. Since there is not complexity of freedom, the BTZ black hole may be taken as a candidate studying the quantum property of the black hole. As the area law is a general property of a black hole, validating the area relation of BTZ black hole should be very important. However, we now can't assert that the area relation is valid in BTZ black hole. The mass entropy of BTZ black hole does not satisfy the area relation just as the result given by (Ichinose and Satoh, 1995). And the geometric structure of the BTZ black hole is different from a usual four-dimensional Schwarzschild black hole.

Recently, (Li, 2002; Zhao *et al.*, 2001; Liu *et al.*, 2003) obtained the new phase volume and calculated the entropy of spherically symmetric black hole via the new equation of state density motivated by the generalized uncertainty relation. The results showed that entropy of black hole only contained the term that is proportional to the area of the horizon. There are not other divergent terms. A new simple and direct way for discussing statistical origin of the black hole is offered.

We generalize the method given by Zhao *et al.* (2001) to investigate the entropy of BTZ black hole and obtain that the entropy of the black line is proportional to the area of the horizon (perimeter). However, using the new equation of state density improved by the generalized uncertainty relation, we only need research in the lamella near the horizon under Planck scale and need not introduce cutoff. Such entropy is the number of quantum state on the surface of black hole horizon. The black line entropy is the intrinsic property of the black hole. The entropy is a quantum effect. By using quantum statistical method (Zhao *et al.*, 2001), we avoid the difficulty to solve wave equation of various particles. We offer a new simple and direct way for calculating the entropy of various spacetime black holes. In this paper, we take the simplest function form of temperature ($C = G = K_B = 1$).

2. BOSONIC ENTROPY

Generalized uncertainty relation (Chang *et al.*, 2002)

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2. \tag{1}$$

According to (1), the minimal uncertainty degree of location is $2\sqrt{\lambda}$ under Planck scale. In $V d^D P$ phase volume, the number of quantum states is given by

$$\frac{V d^D p}{(2\pi\hbar)^D (1 + \lambda p^2)^D}, \tag{2}$$

where λ is the measure of Planck length, D is a dimension factor.

Three-dimensional BTZ black hole metric is given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^{\varphi} dt + d\varphi)^2, \tag{3}$$

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^{\varphi} = -\frac{J}{2r^2}, \quad l^2 = -\frac{1}{\Lambda}, \tag{4}$$

where M and J are the mass, angular momentum, of a black hole, respectively. Λ is the cosmological constant. Equation of horizon can be written as

$$N^2 = \frac{1}{l^2 r^2}(r^2 - r_+^2)(r^2 - r_-^2) = 0, \tag{5}$$

here we define r_+ and r_- are

$$r_\pm = \sqrt{Ml}\left[\frac{1}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right)\right]^{1/2}. \tag{6}$$

Because the nonextreme BTZ black holes satisfy the relation $Ml > J$, r_+ and r_- are the locations of outer event horizon and inner Cauchy horizon respectively. Hawking radiation temperature is

$$T_H = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2}. \tag{7}$$

In the view of the Tolman (1934), the natural radiation temperature got by the observer at r is as follows:

$$T = \frac{T_H}{\sqrt{-\tilde{g}_{tt}}}, \tag{8}$$

where

$$-\tilde{g}_{tt} = -\frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}. \tag{9}$$

For bosonic gas, we calculate the partition function of the system as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}). \tag{10}$$

From (2), in unit area, the density of quantum states is given by

$$g(\nu) = j\frac{1}{(2\pi\hbar)^2}\frac{2\pi p^2}{(1 + \lambda p^2)^2}, \tag{11}$$

where j is the spinning degeneracy of particles. For space-time (3), the area element of a surface $t =$ const is

$$dS = 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr. \tag{12}$$

So the partition function for massless bosons is

$$
\begin{aligned}
\ln Z &= -\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\sum_i g_i\ln(1-e^{\beta\varepsilon_i})\\
&= -\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\int_0^\infty dg(\nu)\ln(1-e^{-\beta h\nu})\\
&= j\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\int_0^\infty \frac{2\pi\beta h}{(1+\lambda 4\pi^2\nu^2)^2(e^{\beta h\nu}-1)}\nu^2 d\nu\\
&= j\beta_0\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\int_0^\infty \frac{2\pi h\sqrt{-\tilde{g}_{tt}}}{(1+\lambda 4\pi^2\nu^2)^2(e^{\beta h\nu}-1)}\nu^2 d\nu, \qquad (13)
\end{aligned}
$$

where $\beta=\beta_0\sqrt{-\tilde{g}_{tt}}$, $T_H=\frac{1}{\beta_0}$ is the radiation temperature of the black hole, j is the spinning degeneracy of radiation particles. According to the relation between free energy and partition function, we have

$$
F=-\frac{1}{\beta_0}\ln Z=-j\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\int_0^\infty \frac{2\pi h\sqrt{-\tilde{g}_{tt}}}{(1+\lambda 4\pi^2\nu^2)^2(e^{\beta h\nu}-1)}\nu^2 d\nu. \qquad (14)
$$

Thus, the entropy of the system can be expressed as

$$
\begin{aligned}
S_b &= \beta_0^2\frac{\partial F}{\partial\beta_0}=j\beta_0\int 2\pi\sqrt{g_{\varphi\varphi}g_{rr}}dr\int_0^\infty \frac{2\pi\beta\nu h^2\sqrt{-\tilde{g}_{tt}}e^{\beta h\nu}}{(1+\lambda 4\pi^2\nu^2)^2(e^{\beta h\nu}-1)^2}\nu^2 d\nu\\
&= j\frac{1}{2\pi\beta_0^2}\int\frac{\sqrt{g_{\varphi\varphi}g_{rr}}}{(-\tilde{g}_{tt})}dr\int_0^\infty \frac{e^x x^3 dx}{\left(1+\lambda\frac{x^2}{\beta_0^2(-\tilde{g}_{tt})}\right)^2(e^x-1)^2}, \qquad (15)
\end{aligned}
$$

where $x=\beta h\nu$, suppose

$$
\begin{aligned}
I_1(\tilde{g}_{tt}) &= \int_0^\infty \frac{e^x x^3 dx}{\left(1+\lambda\frac{x^2}{\beta_0^2(-\tilde{g}_{tt})}\right)^2(e^x-1)^2}\\
&\approx \int_0^\infty \frac{(x+x^2)dx}{\left(1+\lambda\frac{x^2}{\beta_0^2(-\tilde{g}_{tt})}\right)^2}=\frac{\beta_0^2(-\tilde{g}_{tt})}{2\lambda}+\frac{\pi}{4}\beta_0^3\left(\frac{-\tilde{g}_{tt}}{\lambda}\right)^{3/2}. \qquad (16)
\end{aligned}
$$

In (15), In order to calculate the entropy of black hole we integrate near the horizon of the black hole and take the integral region$[r_+, r_+ + \varepsilon]$ with respect to r. Substituting (16) into (15), we obtain

$$S_b = j\frac{1}{2\pi\beta_0^2}\int_{r_+}^{r_++\varepsilon}\frac{\sqrt{g_{\varphi\varphi}g_{rr}}}{(-\tilde{g}_{tt})}dr\left[\frac{\beta_0^2(-\tilde{g}_{tt})}{2\lambda}+\frac{\pi}{4}\beta_0^3\left(\frac{-\tilde{g}_{tt}}{\lambda}\right)^{3/2}\right]$$
$$= j\frac{r_+}{4\pi\lambda}\sqrt{\frac{2\varepsilon}{\kappa}}+j\frac{r_+}{8}\beta_0\frac{\varepsilon}{\lambda^{3/2}}. \tag{17}$$

We are only interested in the contribution from the vicinity near the horizon. According to generalized uncertainty relation (1), we derive that the minimal uncertainty degree is $2\sqrt{\lambda}$ under Planck scale.

Hence, taking as the minimal length of linear element of pure spacetime, $2\sqrt{\lambda}$, has the following form.

$$2\sqrt{\lambda} = \int_{r_+}^{r_++\varepsilon}\sqrt{g_{rr}}dr \approx \int_{r_+}^{r_++\varepsilon}\frac{dr}{\sqrt{2\kappa(r-r_+)}} = \sqrt{\frac{2\varepsilon}{\kappa}}, \tag{18}$$

where κ is the surface gravity at the horizon of black hole and it is identified as $\kappa = 2\pi\beta_0^{-1}$. Thus we naturally derive the expression of entropy

$$S_b = j\frac{A(r_+)}{\lambda_0}, \tag{19}$$

where $\lambda_0 = \frac{1}{4\lambda^{1/2}}\left[\frac{1}{\pi^2}+1\right]$, $A(r_+) = 2\pi r_+$ is the area of the horizon (perimeter).

3. FERMI ENTROPY

For fermi gas, the partition function is as follows:

$$\ln Z = \sum_i g_i \ln(1+e^{-\beta\varepsilon_i}). \tag{20}$$

From (11) and (15), we have

$$S_f = \beta_0^2\frac{\partial F}{\partial\beta_0} = i\frac{1}{2\pi\beta_0^2}\int\frac{\sqrt{g_{\varphi\varphi}g_{rr}}}{(-\tilde{g}_{tt})}dr\int_0^\infty\frac{e^x x^3 dx}{\left(1+\lambda\frac{x^2}{\beta_0^2(-\tilde{g}_{tt})}\right)^2(e^x+1)^2}. \tag{21}$$

Let

$$I_2 = \int_0^\infty\frac{e^x x^3 dx}{\left(1+\lambda\frac{x^2}{\beta_0^2(-\tilde{g}_{tt})}\right)^2(e^x+1)^2} = \int_0^\infty\frac{e^x x^3 dx}{(1+\mu x^2)^2(e^x+1)^2}$$

$$= -\frac{\partial}{\partial\mu}\int_0^\infty \frac{e^x x dx}{(1+\mu x^2)(e^x+1)^2} \approx -\frac{\partial}{\partial\mu}\int_0^\infty \frac{(x+x^2)dx}{(1+\mu x^2)(x+2)^2}$$

$$= -\frac{\partial}{\partial\mu}\int_0^\infty \left[+\frac{3\mu}{4\mu+1}\frac{x}{(\mu x^2+1)} - \frac{3}{4\mu+1}\frac{1}{(x+2)} + \frac{1-2\mu}{4\mu+1}\frac{1}{(\mu x^2+1)} \right.$$
$$\left. + \frac{2}{(x+2)^2(1+\mu x^2)} \right] dx$$

$$\approx \frac{\partial}{\partial\mu}\left[\frac{3}{2(4\mu+1)}\ln(1+\mu x^2) - \frac{3}{4\mu+1}\ln(x+2) + \frac{1-2\mu}{4\mu+1}\frac{1}{\sqrt{\mu}} arctgx\sqrt{\mu} \right]_0^\infty$$
$$-\frac{\partial}{\partial\mu}\left[\frac{1}{4(\mu+1)}\ln(1+\mu x^2) - \frac{1}{2\mu+1}\ln(x+1) - \frac{\sqrt{\mu}}{2(\mu+1)} arctgx\sqrt{\mu} \right]_0^\infty$$
$$\approx \frac{1}{16}\mu^{-2} = \frac{1}{16}\beta_0^4\left(\frac{-\tilde{g}_{tt}}{\lambda}\right)^2, \tag{22}$$

we obtain the entropy of Fermi field

$$S_f = i\int_{r_+}^{r_+ + \varepsilon} \frac{r\beta_0^2}{32\pi}\frac{\sqrt{-\tilde{g}_{tt}}}{\lambda^2} dr = i\frac{2}{3}\frac{A(r_+)}{\lambda^{1/2}}, \tag{23}$$

where i is the spinning degeneracy of fermions.

4. CONCLUSION

Based on the above analysis, by using statistical method, we directly obtain the partition function of various fields on the background of BTZ black hole and avoid the difficult to solve wave equations. In our calculation, using the new equation of state density motivated by the generalized uncertainty relation, we need not introduce cutoff. The problem why the entropy of the radiation field outside the horizon is the entropy of the black hole is solved. There are not the left out term and the divergent logarithmic term in the original brick-wall method. The method to calculate entropies of black hole via the generalized uncertainty relation is valid not only for four-dimensional spacetimes but also for three-dimensional spacetimes. Thus our methods have universality.

As early as 1992, Li and Liu phenomenally proposed the state equations motivated by gravity and gave the state equations of the thermal radiation filed near the horizon of black hole (Li and Liu, 1992). Using the Li-Liu equation, Wang calculated the entropy of a black hole and obtained that the entropy of the black hole is proportional to the area of the horizon (Wang, 1994). Moreover, in his calculation the left out term and the divergent logarithmic term in the original brick-wall method don't exist. In this paper, through investigating BTZ black hole, we also obtain that the entropy of the black line is proportional to the area of the horizon in the case that there is not the divergent term. We start with different consideration, but obtain the same conclusion. There is an inherent relationship between the Li-Liu equation and generalized uncertainty relation not only in four-dimensional spacetime but also in lower-dimensional spacetime. So the inherent relationship between the Li-Liu equation and generalized uncertainty relation is a general problem. Solving this problem is an important subject of theoretic physics.

ACKNOWLEDGMENT

This project was supported by the National Natural Science Foundation of China under Grant No. 10374075 and the Shanxi Natural Science Foundation of China (No 2006011012).

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